Problem 2 (parts b,c,d). Let $A_1, A_2, \dots \subseteq S$ be events. Define $B_1 := A_1$ and for $i > 1$,

$$B_i := A_i \cap \Big(\bigcup_{j=1}^{i-1} A_j \Big)^c.$$

(b) Show that the B_i are pairwise disjoint.

Proof. Fix $i < k$. By definition,

$$B_k \subseteq \Big(\bigcup_{j=1}^{k-1} A_j \Big)^c \subseteq A_i^c,$$

since $A_i \subseteq \bigcup_{j=1}^{k-1} A_j$ when $i \le k-1$. Also $B_i \subseteq A_i$. Therefore

$$B_i \cap B_k \subseteq A_i \cap A_i^c = \varnothing.$$

Hence $B_i \cap B_k = \varnothing$ for all $i \neq k$, so the B_i are pairwise disjoint. $\square$

(c) Show that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$.

Proof. First, for each i we have $B_i \subseteq A_i$, so

$$\bigcup_{i=1}^{\infty} B_i \subseteq \bigcup_{i=1}^{\infty} A_i.$$

For the reverse inclusion, take any $x \in \bigcup_{i=1}^{\infty} A_i$. Let m be the *smallest* index such that $x \in A_m$. Then $x \notin A_j$ for all $j < m$, so $x \in \big(\bigcup_{j=1}^{m-1} A_j \big)^c$. Since also $x \in A_m$, we have

$$x \in A_m \cap \Big(\bigcup_{j=1}^{m-1} A_j \Big)^c = B_m \subseteq \bigcup_{i=1}^{\infty} B_i.$$

Thus $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} B_i$, completing the proof. $\square$

Alternate proof of (c) via induction. We prove by induction on n that

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i \qquad \text{for all } n \ge 1.$$

Base case ($n = 1$). Since $B_1 = A_1$, we have $\bigcup_{i=1}^{1} A_i = A_1 = B_1 = \bigcup_{i=1}^{1} B_i$.

Inductive step. Assume for some $n \geq 1$ that $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$. We show the statement holds for $n+1$.

First note that $B_{n+1} \subseteq A_{n+1}$, so

$$\bigcup_{i=1}^{n+1} B_i = \Big(\bigcup_{i=1}^{n} B_i\Big) \cup B_{n+1} \subseteq \Big(\bigcup_{i=1}^{n} A_i\Big) \cup A_{n+1} = \bigcup_{i=1}^{n+1} A_i,$$

where we used the inductive hypothesis in the middle inclusion.

For the reverse inclusion, take $x \in \bigcup_{i=1}^{n+1} A_i$. If $x \in \bigcup_{i=1}^{n} A_i$, then by the inductive hypothesis $x \in \bigcup_{i=1}^{n} B_i \subseteq \bigcup_{i=1}^{n+1} B_i$. Otherwise, $x \in A_{n+1}$ and $x \notin \bigcup_{i=1}^{n} A_i$. Equivalently, $x \in \big(\bigcup_{i=1}^{n} A_i\big)^c$, hence

$$x \in A_{n+1} \cap \Big(\bigcup_{i=1}^{n} A_i\Big)^c = B_{n+1} \subseteq \bigcup_{i=1}^{n+1} B_i.$$

Thus $\bigcup_{i=1}^{n+1} A_i \subseteq \bigcup_{i=1}^{n+1} B_i$.

Combining the two inclusions yields $\bigcup_{i=1}^{n+1} A_i = \bigcup_{i=1}^{n+1} B_i$, completing the inductive step. Therefore $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$ for all n.

Finally, taking unions over n gives

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{n=1}^{\infty}\bigcup_{i=1}^{n} A_i = \bigcup_{n=1}^{\infty}\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{\infty} B_i,$$

as desired. □

(d) Use the previous parts to show that $\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

Proof. By part (c),

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \mathbb{P}\Big(\bigcup_{i=1}^{\infty} B_i\Big).$$

By part (b), the events B_i are pairwise disjoint, so by countable additivity,

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \mathbb{P}(B_i).$$

Finally, since $B_i \subseteq A_i$ for each i, monotonicity gives $\mathbb{P}(B_i) \le \mathbb{P}(A_i)$, hence

$$\sum_{i=1}^{\infty} \mathbb{P}(B_i) \le \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Combining the displays yields

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) \le \sum_{i=1}^{\infty} \mathbb{P}(A_i),$$

as desired. $\square$