Lab0: Problem Set 0 Notes

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Problem (1)

Explain why this is horrific notation:

$$\int_0^x f(x)dx.$$

Try this out yourselves!

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Note: Look at each piece of the integral carefully – if you're having trouble finding the notational misstep, you have subconsciously identified and "autocorrected" it in your mind.

Problem (2)

Simplify this:

$$\ln\left(e^{a_1}\cdots e^{a_n}\right).$$

Try this out yourselves!

Problem (3)

Assume $\lambda > 0$ is a constant and compute

$$\sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda}.$$

We will be handling **Taylor series** in this problem.

Taylor Series Review

Let $a \in \mathbb{C}$ and f be a function infinitely differentiable at a. The Taylor series expansion of f about a is given in general by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $f^{(n)}(a)$ is the nth derivative of f evaluated at a.

When $a = 0$, this is called the Maclaurin series and is of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

In particular, we will be using the Taylor (Maclaurin) series for e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{\frac{d^n e^x(0)}{dx^n}}{n!}x^n = \sum_{n=0}^{\infty} \frac{e^0}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Problem (3) Work

$$
\begin{aligned}
\sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} &= e^{-\lambda} \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \left(0 \cdot \underbrace{\frac{\lambda^0}{0!}}_{=\frac{1}{1}} + \sum_{n=1}^{\infty} n \frac{\lambda^n}{n!} \right) \\
&= e^{-\lambda} \sum_{n=1}^{\infty} n \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \sum_{n=1}^{\infty} n \frac{\lambda \cdot \lambda^{n-1}}{n \cdot (n-1)!} \\
&= \lambda \cdot e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}
\end{aligned}
$$

Problem (3) Work, continued

Let $m = n - 1$. Perform a change of variables:

$$\begin{aligned}
\sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} &= \lambda \cdot e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \\
&= \lambda \cdot e^{-\lambda} \underbrace{\sum_{m=0}^{\infty} \frac{\lambda^m}{m!}}_{=e^{\lambda}} \\
&= \lambda \cdot e^{-\lambda} e^{\lambda} \\
&= \lambda.
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} = \lambda. \quad \square$$

Problem (4)

Here is a very silly function:

$$h(x) = \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \quad -\infty < x < \infty.$$

Treat $-\infty < \mu < \infty$ and $\sigma > 0$ as constants and compute the value(s) of x at which h has *inflection points*.

A solid foundation will be laid for this problem, after which the rest will be up to you!

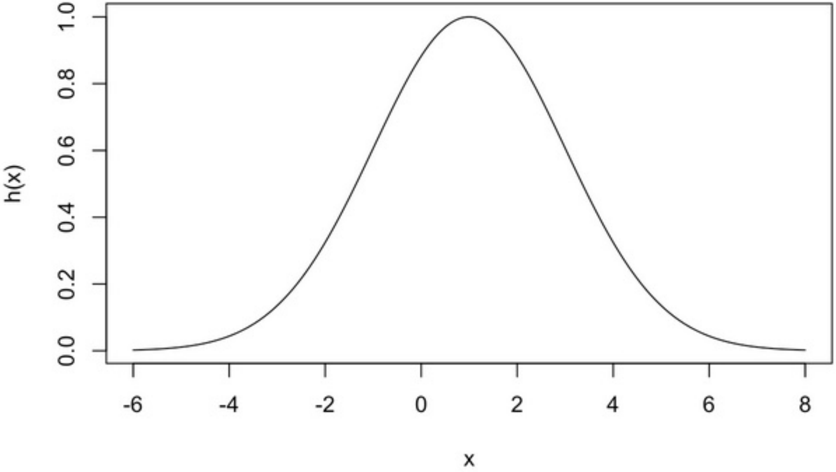
Inflection Point Review

Recall that the **concavity** of a function is a statement on how that function's graph opens or curves. If a function is concave *up*, the graph's curvature resembles $\bigcup$; if a function is concave *down*, the graph's curvature resembles $\bigcap$.

While some functions, like quadratics ($f(x) = ax^2 + bx + c$), are completely concave up or down, other functions have changing concavity. An **inflection point** is a point at which this change occurs.

Inflection Point Review

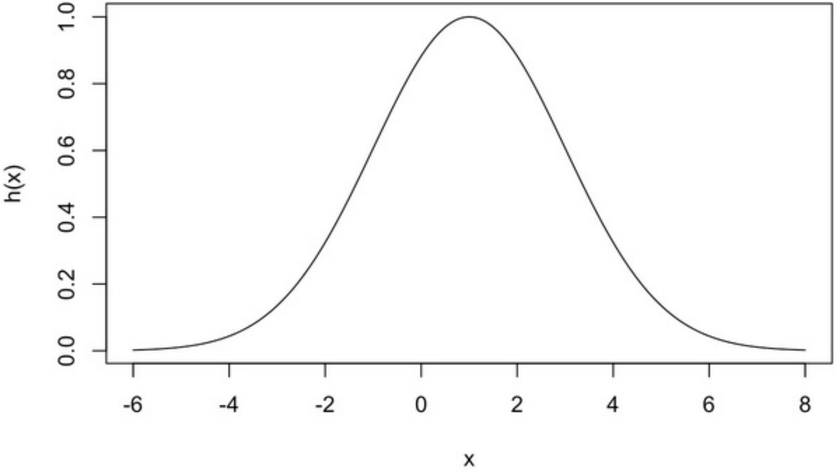
Consider the following graph of $h(x)$ with $\mu = 1$ and $\sigma = 2$:



Observe that around $x = -1$, h seems to change from concave up to concave down.

Inflection Point Review

Consider the following graph of $h(x)$ with $\mu = 1$ and $\sigma = 2$:



In fact, these changes occur **precisely** at $x = -1$ and $x = 3$.

Inflection Point Review

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Concavity changes when the sign of the second derivative changes; as such, for a nicely behaved function f, x_0 is an inflection point if $f''(x_0) = 0$.

Derivative Review

Derivative Properties and Rules

Let f and g be differentiable functions, and let a and c be constants.

1. Linearity: We have that

$$\frac{d}{dx}\left[a \cdot f(x) + c \cdot g(x)\right] = a \cdot f'(x) + c \cdot g'(x).$$

2. Chain Rule: We have that

$$\frac{d}{dx}\left[f(g(x))\right] = g'(x) f'(g(x)).$$

3. Product Rule: We have that

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x).$$

Problem (4) Work

We work through the first derivative together.

$$
\begin{aligned}
h(x) &= \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) \\
\Rightarrow h'(x) &= \frac{d}{dx}\left[\exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)\right] \\
&= \frac{d}{dx}\left[-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right]\exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) && \text{(chain rule)} \\
&= -\frac{1}{2\sigma^2}\frac{d}{dx}\left[(x-\mu)^2\right]\exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) && \text{(linearity)} \\
&= -\frac{1}{2\sigma^2}((1)(2)(x-\mu))\exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) && \text{(chain rule)} \\
&= -\frac{x-\mu}{\sigma^2}\cdot\exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)
\end{aligned}
$$

You do the rest! The properties and rules (click here) may be of use.

Problem (5)

Here is another inordinately silly function:

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy, \quad x > 0.$$

Prove that $\Gamma(x+1) = x\Gamma(x)$.[1]

In order to work through this, we will review **integration by parts** and **improper integration**.

[1]Observe that, when $x \in \{1, 2, 3, \ldots\}$, $\Gamma(x) = (x-1)!$

Integration Review

Consider the definite integral $\int_a^b f(x)dx$. We have that a and b are the *lower* and *upper limits of integration*, respectively, x is the *variable of integration*, and $f(x)$ is the integrand (i.e., the function that is to be integrated).

Unfortunately, not every integrand has a nice, closed-form antiderivate; in Fundamental Theorem of Calculus terms, this means that there does not necessarily exist an F such that

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

However, in certain circumstances, we can still work with unwieldy-looking integrands.

Integration by Parts Review

Recall that the Product Rule for differentiation states that, for differentiable functions f and g,

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x).$$

Recall that differentiation and integration are inverses of each other. So, rearranging terms and integrating both sides (and ignoring $+C$ terms for indefinite integrals), we have

$$\begin{aligned} \frac{d}{dx}\left[f(x)g(x)\right] &= f'(x)g(x) + f(x)g'(x) \\ \Rightarrow f(x)g'(x) &= \frac{d}{dx}\left[f(x)g(x)\right] - f'(x)g(x) \\ \Rightarrow \int f(x)g'(x)dx &= \int \frac{d}{dx}\left[f(x)g(x)\right]dx - \int f'(x)g(x)dx \\ \Rightarrow \int f(x)g'(x)dx &= f(x)g(x) - \int f'(x)g(x)dx. \end{aligned}$$

Integration by Parts Review

This is the basis for integration by parts. Essentially, the integrand of interest is one term from the result of applying the Product Rule for two functions. We try to evaluate (or at least rewrite) that term using an expedient combination of derivatives and anti-derivatives.

The general integration by parts framework is typically given with the following notation:

$$\int u dv = uv - \int v du.$$

Problem (5) Work

We have that $x > 0$ is a constant and consider $\Gamma(x+1)$; we have that

$$\Gamma(x+1) = \int_0^\infty y^{(x+1)-1} e^{-y} dy = \int_0^\infty y^x e^{-y} dy.$$

We want to identify u and dv (thereby identifying v); if the integrand of interest is the product of a power and an exponential, it will generally helpful to identify the power as u and the exponential as dv. So, we have the following:

$$u = y^x \qquad du = xy^{x-1} dy$$

$$v = -e^{-y} \qquad dv = e^{-y} dy.$$

Problem (5) Work

So, we proceed as follows:[2]

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty y^x e^{-y} dy \\
&= -y^x e^{-y}\Big|_0^\infty - \int_0^\infty \left(-e^{-y}\right)\left(xy^{x-1}\right) dy \\
&= -y^x e^{-y}\Big|_0^\infty + \int_0^\infty x e^{-y} y^{x-1} dy \\
&= -y^x e^{-y}\Big|_0^\infty + x \int_0^\infty y^{x-1} e^{-y} dy
\end{aligned}$$

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[2]Note: For a function $f(y)$, $f(y)|_a^b = f(y)|_{y=b} - f(y)|_{y=a} = f(b) - f(a)$.

Improper Integration Review

Suppose you want to evaluate an integral $\int_a^b f(x)dx$, but $a = -\infty$, $b = \infty$, or f is discontinuous at some point between a and b, inclusive. This means the integral is *improper*.

Consider the case where the integral is of the form $\int_a^\infty f(x)dx$, with $-\infty < a < \infty$. To evaluate the integral, you take the following limit:

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

If f has an antiderivative F, this is equivalent to

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx = \lim_{b\to\infty} F(x)\big|_a^b = \lim_{b\to\infty} F(b) - F(a).$$

Problem (5) Work, Continued

The same principle applies to evaluating a function "at $\pm\infty$." It is not strictly proper to "plug-in" $\pm\infty$; rather, the appropriate limit should be taken.

Consider the $-y^x e^{-y}\big|_0^\infty = -\frac{y^x}{e^y}\Big|_0^\infty$ term, recalling that $x > 0$ is a constant. Evaluating at 0 is simple enough:

$$-\frac{y^x}{e^y}\bigg|_{y=0} = -\frac{0^x}{e^0} = -\frac{0}{1} = 0.$$

In order to "evaluate" at ∞, we need to take the following limit:

$$-\frac{y^x}{e^y}\bigg|_{y=\infty} = \lim_{y\to\infty} -\frac{y^x}{e^y}.$$

Problem (5) Work, Continued

Since both the numerator and denominator are going to $(\pm)\infty$, we cannot evaluate the limit in this form. However, intuitively, the exponential function e^y with blow up to ∞ *much faster* than a power function like y^x. So,

$$\lim_{y\to\infty} -y^x e^{-y} = 0.$$

A discussion of L'Hôpital's Rule (click here) is forthcoming, as is a more mathematically rigorous evaluation of this limit (click here).

Problem (5) Work, Continued

We thus have

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty y^x e^{-y} dy \\ &= -y^x e^{-y}\Big|_0^\infty + x\int_0^\infty y^{x-1} e^{-y} dy \\ &= x\int_0^\infty y^{x-1} e^{-y} dy.\end{aligned}$$

Take the final step to prove the original statement!

Problem (6)

Let f be any function with the following properties:

- f is twice continuously differentiable in a neighborhood of 0;
- $f(0) = 0$;
- $f'(0) = 0$;
- $f''(0) = 1$.

Assume t is a constant and compute

$$\lim_{x \to \infty} x f\left(\frac{t}{\sqrt{x}}\right).$$

To work through this, we will review **continuity**, **differentiability**, and **L'Hôpital's Rule**.

Continuity and Differentiability Review

Recall that a function f is continuous at a point x_0 if

$$\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right) = f\left(x_0\right).$$

f is differentiable at x_0 if $f'\left(x_0\right)$ is well-defined.

f is continuously differentiable in a neighborhood of x_0 if, on some open interval containing x_0, f' is well-defined and continuous.

f is twice continuously differentiable in a neighborhood of x_0 if, on some open interval containing x_0, f'' is also well-defined and continuous.

Recall that a function that is differentiable on an interval is also continuous on that interval.

Problem (6) Work

We have that t is a constant and proceed as follows:

$$\begin{aligned}
\lim_{x\to\infty} xf\left(\frac{t}{\sqrt{x}}\right) &= \left(\lim_{x\to\infty} x\right)\left(\lim_{x\to\infty} f\left(\frac{t}{\sqrt{x}}\right)\right) && \text{(product of limits)} \\
&= \left(\lim_{x\to\infty} x\right)\cdot f\left(\lim_{x\to\infty} \frac{t}{\sqrt{x}}\right) && \text{continuity of } f \\
&= \infty \cdot f\left(t \lim_{x\to\infty} \frac{1}{\sqrt{x}}\right) \\
&= \infty \cdot f(t\cdot 0) \\
&= \infty \cdot f(0) \\
&= \infty \cdot 0.
\end{aligned}$$

$\infty \cdot 0$ is a problematic (non-)answer. So, what do we do?

L'Hôpital's Rule Review

If you are taking the limit of a function and the limit looks like $\pm\frac{\infty}{\infty}, \frac{0}{0}, \infty \cdot 0$, or a few others, you have encountered an *indeterminate form*.

L'Hôpital's Rule

Let f and g be two functions defined on an open interval containing at point c and differentiable on that interval (except, perhaps, at c). If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \pm\infty$ or 0, g is non-zero in that open interval (expect, perhaps, at c), and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Essentially, if you encounter an indeterminate form and can rewrite the function whose limit you're trying to find as a quotient of two functions, use L'Hôpital's Rule!

Problem (6) Work, Continued

We have the following:

$$\begin{aligned}
\lim_{x\to\infty} xf\left(\frac{t}{\sqrt{x}}\right) &= \lim_{x\to\infty} \frac{f\left(tx^{-1/2}\right)}{x^{-1}} && \left(x = \frac{1}{1/x} = \frac{1}{x^{-1}}\right) \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}\left[f\left(tx^{-1/2}\right)\right]}{\frac{d}{dx}\left[x^{-1}\right]} && \text{(L'Hôpital's rule)} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}\left[tx^{-1/2}\right] f'\left(tx^{-1/2}\right)}{-x^{-2}} && \text{(chain rule)} \\
&= \lim_{x\to\infty} \frac{-\frac{t}{2}x^{-3/2} f'\left(tx^{-1/2}\right)}{-x^{-2}} \\
&= \lim_{x\to\infty} \frac{-\frac{t}{2} f'\left(tx^{-1/2}\right)}{-x^{-1/2}} \\
&= \lim_{x\to\infty} \frac{t}{2}x^{1/2} f'\left(\frac{t}{\sqrt{x}}\right).
\end{aligned}$$

Complete the rest of the problem!

Problem (7)

Consider this integral:

$$\int_2^\infty \frac{1}{x(\ln x)^p} dx.$$

1. Use a computer to create a single plot with many lines, each graphing the *integrand* for a different value of p. Consider p equal to -2, -1.5, -1, 0, 1, and 5, and make the x-axis of your plot run from 2 to 15.
2. Show that $\lim_{x\to\infty} \frac{1}{x(\ln x)^p} = 0$ for all values of $-\infty < p < \infty$.
3. For what values of p does the integral converge? When it does converge, what is its value?
4. Consult the picture you created in part (a), and write a few sentences explaining conceptually why the integral converges for some values of p but not others.

Problem (7) Advice

Try this out yourselves! Here are some helpful reminders/tips:

- Remember that, in the integral $\int_a^b f(x)dx$, $f(x)$ is the integrand.
- For part (a), use any software you would like (R, Excel, online graphing calculators, etc.) – just make sure a screenshot or image makes its way into your final PDF submission.
- For part (b), case work may be useful. A proof for some values of p may not work so well for other values of p.

We will work through the $p \geq 0$ case for part (b) together.

Problem (7) Work

Suppose $p \geq 0$. Consider the behavior of the integrand for all $x \geq 3$:

$$0 \leq \frac{1}{x(\ln x)^p} \leq \frac{1}{x}.$$

By the squeeze theorem, we observe the following:

$$\begin{aligned} 0 &\leq \frac{1}{x(\ln x)^p} \leq \frac{1}{x} \\ \Rightarrow \lim_{x\to\infty} 0 &\leq \lim_{x\to\infty} \frac{1}{x(\ln x)^p} \leq \lim_{x\to\infty} \frac{1}{x} \\ \Rightarrow 0 &\leq \lim_{x\to\infty} \frac{1}{x(\ln x)^p} \leq 0. \end{aligned}$$

Therefore, for $p \geq 0$, we have that

$$\lim_{x\to\infty} \frac{1}{x(\ln x)^p} = 0.$$

Additional Problem 7 Advice

To work through the $p < 0$ case of part (b), try to rewrite the integrand in an easier-to-handle form. (Perhaps you will be able to use something we've encountered earlier in this problem set!)

Addendum: Problem (5) L'Hôpital's Rule

We consider two cases. First, suppose $x \in \mathbb{N}$; that is, $x \in \{1, 2, 3, \dots\}$. Applying L'Hôpital's Rule x times yields the following:

$$\begin{aligned}
\lim_{y\to\infty} -\frac{y^x}{e^y} &= \lim_{y\to\infty} -\frac{\frac{d^x}{dy^x}\left[y^x\right]}{\frac{d^x}{dy^x}\left[e^y\right]} \\
&= \lim_{y\to\infty} -\frac{x(x-1)\cdots 2\cdot 1\cdot y^0}{e^y} \\
&= \lim_{y\to\infty} -\frac{x(x-1)\cdots 2\cdot 1\cdot 1}{e^y} \\
&= -x(x-1)\cdots 2 \lim_{y\to\infty} \frac{1}{e^y} \\
&= -x(x-1)\cdots 2\cdot 0 \\
&= 0.
\end{aligned}$$

Addendum: Problem (5) L'Hôpital's Rule

Now suppose $x \notin \mathbb{N}$ and recall that $\lfloor x \rfloor$ rounds x *down* to the nearest integer. There is some $0 < c < 1$ such that $x = \lfloor x \rfloor + c$. Applying L'Hôpital's Rule $\lfloor x \rfloor + 1$ times yields:

$$\begin{aligned}
\lim_{y\to\infty} -\frac{y^x}{e^y} &= \lim_{y\to\infty} -\frac{\frac{d^{\lfloor x \rfloor+1}}{dy^{\lfloor x \rfloor+1}}\left[y^x\right]}{\frac{d^{\lfloor x \rfloor+1}}{dy^{\lfloor x \rfloor+1}}\left[e^y\right]} \\
&= \lim_{y\to\infty} -\frac{x(x-1)\cdots(x-\lfloor x \rfloor)\cdot y^{x-(\lfloor x \rfloor+1)}}{e^y} \\
&= \lim_{y\to\infty} -\frac{x(x-1)\cdots(c)\cdot y^{c-1}}{e^y} \\
&= \lim_{y\to\infty} -\frac{x(x-1)\cdots(c)}{y^{1-c}e^y} \\
&= -x(x-1)\cdots(c)\lim_{y\to\infty}\frac{1}{y^{1-c}e^y} \\
&= 0.
\end{aligned}$$