STA 332/MATH 343: Homework 1

Duke University, Fall 2026

Due: September 6th, 11:59 PM on Gradescope

Theory

1. Briefly define the notion of statistic.
2. Consider the statistical model $\{\mathcal{N}(\mu, \sigma^2) \mid \mu \in \mathbb{R}\}$, with σ^2 known, and let $\underline{\mathrm{X}} = (X_1, \ldots, X_n)$ be a random sample. What is the parameter space? For each of the following functions write whether they are statistics or not and why.

$$T_1(\underline{\mathrm{X}}) = \sum_{i=1}^{n} X_i^2, \quad T_2(\underline{\mathrm{X}}) = X_3 + 1, \quad T_3(\underline{\mathrm{X}}) = \sum_{i=1}^{n} X_i^4 + \sigma^2, \quad T_4(\underline{\mathrm{X}}) = \mu e^{X_n}.$$

3. Consider the statistical model $\{\mathcal{N}(\mu, \sigma^2) \mid \mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}_+\}$ and let $\underline{\mathrm{X}} = (X_1, \ldots, X_n)$ be a random sample. For each of the following functions write whether they are statistics or not and why.

$$T_1(\underline{\mathrm{X}}) = \sum_{i=1}^{n} X_i^2, \quad T_2(\underline{\mathrm{X}}) = X_3 + 1, \quad T_3(\underline{\mathrm{X}}) = \sum_{i=1}^{n} X_i^4 + \sigma^2, \quad T_4(\underline{\mathrm{X}}) = \mu e^{X_n}.$$

Probability Recap

4. Let $X \sim \mathsf{Poisson}(\lambda)$. Compute $\mathrm{Var}(X)$. *(Hint: note that $x^2 = x(x+1-1) = x(x-1) + x$).*
5. Let $X \sim \mathsf{Beta}(\alpha, \beta)$, whose density function is given by

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{(0,1)}(x),$$

   where $\alpha, \beta > 0$ and $\Gamma(\cdot)$ is the Gamma function. Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

6. Suppose X has density function $f_X(x)$ supported on $\mathbb{R}$.
   (a) Derive the density function of $Y = \mu + \sigma X$, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants.
   (b) Use this to prove that if $X \sim \mathcal{N}(0, 1)$, then $\mu + \sigma X \sim \mathcal{N}(\mu, \sigma^2)$.
7. Let $X \sim \mathsf{Gamma}(\alpha, \beta)$. Find the density function of $Y = \frac{1}{X}$.
8. Let $X \sim \mathcal{N}(0, 1)$. Find the density function of $Y = |X|$ and compute $\mathbb{E}[Y]$. *(Hint: the transformation is not invertible)*
9. Consider the probability density function of a random variable X,

$$f_X(x) = c|x|\mathbb{1}_{\{-2 \le x \le 2\}}.$$

   (a) Find the value of c. *(Hint: the density function needs to integrate to 1).*

(b) Compute $\mathbb{P}(X > 1)$.

10. Let $X_i \overset{\text{i.i.d.}}{\sim} \mathsf{Unif}(0,1)$ and let $Y = \min\{X_1, \dots, X_n\}$.

    (a) Find the density function of Y. *(Hint: try to compute $\mathbb{P}(Y \geq y)$)*

    (b) Compute $\mathbb{E}[Y]$.

11. Consider the following bivariate density function of the vector (X, Y)

$$f(x,y) = \begin{cases} x + y, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{else} \end{cases}$$

    (a) Compute the marginal density functions $f_X(x)$ and $f_Y(y)$.

    (b) Compute $\mathbb{P}\big(X \leq \frac{1}{2} \mid Y = \frac{1}{2}\big)$ and $\mathbb{P}\big(X \leq \frac{1}{2}, Y = \frac{1}{2}\big)$.

Ungraded bonus problems

12. Let $X \sim \mathsf{Gamma}(\alpha, \beta)$ and let $\lambda \in \mathbb{R}$ be a constant.

    (a) Compute $\mathbb{E}\big[e^{\lambda X}\big]$. For which values of λ is the expected value finite?

    (b) Let $X \sim \mathsf{Gamma}(\alpha_1, \beta)$ and $Y \sim \mathsf{Gamma}(\alpha_2, \beta)$ and assume they are independent. Find $\mathbb{E}\big[e^{\lambda(X+Y)}\big]$ when it is finite.

13. Let $X_1 \sim \mathsf{Poisson}(\lambda_1), X_2 \sim \mathsf{Poisson}(\lambda_2)$ be independent random variables.

    (a) Prove that $X_1 + X_2 \sim \mathsf{Poisson}(\lambda_1 + \lambda_2)$.

    (b) Prove that $X_1 \mid X_1 + X_2 = n \sim \mathsf{Bin}\Big(n, \frac{\lambda_1}{\lambda_1 + \lambda_2}\Big)$.