STA 332/MATH 343: Homework 5 – Solutions

Duke University, Fall 2026

Due: Monday, October 19 at 11:59 PM on Gradescope

Fisher Information and the Cramér–Rao Lower Bound

1. Concepts relating to Fisher information in a regular one-parameter model:

   (a) Define the score function.

   (b) Give the two formulas for Fisher information.

   (c) State the Cramér–Rao lower bound for an unbiased estimator of $g(\theta)$ based on n i.i.d. observations.

   (d) Define $\text{eff}(T)$ for an unbiased estimator T, and what it means for an unbiased estimator to be efficient.

   (e) Explain why an unbiased efficient estimator is automatically the UMVUE.

   (a) The score is
   $$S_\theta(x) = \frac{\partial}{\partial\theta} \log f_\theta(x).$$

   (b) For a regular model, the score has mean zero and
   $$I_X(\theta) = \mathbb{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\log f_\theta(X)\right)^2\right] = -\mathbb{E}_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(X)\right].$$
   These identities require the regularity conditions that justify differentiation under the integral or sum.

   (c) For an unbiased estimator T of a differentiable $g(\theta)$ with finite variance, under the required regularity conditions and with $0 < I_{\underline{X}}(\theta) < \infty$,
   $$\text{Var}_\theta(T) \geq \ell_{CR}(\theta) = \frac{[g'(\theta)]^2}{I_{\underline{X}}(\theta)} = \frac{[g'(\theta)]^2}{nI_{X_1}(\theta)}.$$

   (d) When $\text{Var}_\theta(T) > 0$, its efficiency is
   $$\text{eff}_\theta(T) = \frac{\ell_{CR}(\theta)}{\text{Var}_\theta(T)}.$$
   An unbiased estimator is efficient if it attains the Cramér–Rao bound for every θ, i.e. $\text{Var}_\theta(T) = \ell_{CR}(\theta)$ for every θ.

   (e) If T is unbiased and efficient, then any other unbiased estimator U of the same $g(\theta)$ satisfies
   $$\text{Var}_\theta(U) \geq \ell_{CR}(\theta) = \text{Var}_\theta(T) \qquad \text{for every } \theta.$$
   Therefore T is the UMVUE.

2. Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ is unknown and $\sigma^2 > 0$ is known.

   (a) Compute the score for one observation and verify that its mean is zero.

   (b) Compute $I_{\underline{X}}(\mu)$, i.e. the Fisher information of $\underline{X}$.

   (c) Compute the mean and variance of $T = \bar{X}_n$.

   (d) Use the Cramér–Rao bound to prove that T is the UMVUE of μ.

   (e) Without doing any work, what is $I_T(\mu)$ and why?

   (a) The density of one observation is

$$f_\mu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \qquad x \in \mathbb{R}.$$

   Taking logarithms gives

$$\log f_\mu(x) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2}.$$

   Differentiating with respect to μ, treating σ^2 as known, gives

$$S_\mu(x) = -\frac{1}{2\sigma^2}\big[-2(x-\mu)\big] = \frac{x-\mu}{\sigma^2}.$$

   Hence $\mathbb{E}_\mu[S_\mu(X_1)] = (\mathbb{E}_\mu[X_1] - \mu)/\sigma^2 = 0$.

   (b) For an observed sample $\underline{x} = (x_1, \ldots, x_n)$, independence gives the likelihood

$$L(\mu; \underline{x}) = \prod_{i=1}^n f_\mu(x_i) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2\right\}.$$

   Thus the sample log likelihood and its derivatives are

$$\ell(\mu; \underline{x}) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2,$$

$$\ell'(\mu; \underline{x}) = \frac{1}{\sigma^2}\sum_{i=1}^n (x_i - \mu),$$

$$\ell''(\mu; \underline{x}) = -\frac{n}{\sigma^2}.$$

   Therefore

$$I_{\underline{X}}(\mu) = -\mathbb{E}_\mu[\ell''(\mu; \underline{X})] = -\mathbb{E}_\mu\left[-\frac{n}{\sigma^2}\right] = \frac{n}{\sigma^2}.$$

   (c) Independence gives

$$\mathbb{E}_\mu[T] = \frac{1}{n}\sum_i \mathbb{E}_\mu[X_i] = \mu, \qquad \mathrm{Var}_\mu(T) = \frac{1}{n^2}\sum_i \mathrm{Var}_\mu(X_i) = \frac{\sigma^2}{n}.$$

   (d) For estimating $g(\mu) = \mu$, the bound is $1/I_{\underline{X}}(\mu) = \sigma^2/n$. The unbiased estimator T attains it for every μ, so it is the UMVUE.

   (e) The sample mean T is sufficient for μ when σ^2 is known. Hence it retains all the sample Fisher information:

$$I_T(\mu) = I_{\underline{X}}(\mu) = \frac{n}{\sigma^2}.$$

3. Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathsf{Ber}(\theta)$, where $0 < \theta < 1$.

   (a) Compute $I_{\underline{X}}(\theta)$, i.e. the Fisher information of $\underline{X}$.

   (b) For $T = \bar{X}_n$, compute $\mathbb{E}_\theta[T]$ and $\mathrm{Var}_\theta(T)$.

   (c) Use the Cramér–Rao bound to prove that T is the UMVUE of θ.

(a) The pmf of one observation is

$$f_\theta(x) = \theta^x (1-\theta)^{1-x}, \qquad x \in \{0, 1\}.$$

For $\underline{x} \in \{0,1\}^n$, independence gives

$$L(\theta; \underline{x}) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum_i x_i}(1-\theta)^{n-\sum_i x_i}.$$

Taking logarithms and differentiating gives

$$\begin{aligned}
\ell(\theta; \underline{x}) &= \left(\sum_i x_i\right)\log\theta + \left(n - \sum_i x_i\right)\log(1-\theta), \\
\ell'(\theta; \underline{x}) &= \frac{\sum_i x_i}{\theta} - \frac{n - \sum_i x_i}{1-\theta}, \\
\ell''(\theta; \underline{x}) &= -\frac{\sum_i x_i}{\theta^2} - \frac{n - \sum_i x_i}{(1-\theta)^2}.
\end{aligned}$$

Since $\mathbb{E}_\theta[\sum_i X_i] = n\theta$, the Fisher information is

$$\begin{aligned}
I_{\underline{X}}(\theta) &= -\mathbb{E}_\theta[\ell''(\theta; \underline{X})] \\
&= \frac{n\theta}{\theta^2} + \frac{n - n\theta}{(1-\theta)^2} \\
&= n\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right) = \frac{n}{\theta(1-\theta)}.
\end{aligned}$$

(b) Independence gives

$$\mathbb{E}_\theta[T] = \theta, \qquad \mathrm{Var}_\theta(T) = \frac{\theta(1-\theta)}{n}.$$

(c) The bound for unbiased estimation of θ is $1/I_{\underline{X}}(\theta) = \theta(1-\theta)/n$. The variance of T equals this bound for every $0 < \theta < 1$, so T is the UMVUE.

4. Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \tau)$, where μ is known and the variance $\tau > 0$ is unknown.

   (a) Compute $I_{\underline{X}}(\tau)$.

   (b) For $U = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$, compute $\mathbb{E}_\tau[U]$ and $\mathrm{Var}_\tau(U)$.
   Hint: if $Z \sim \mathcal{N}(0,1)$ *then* $\mathbb{E}[Z^4] = 3$.

   (c) Use the Cramér–Rao bound to prove that U is the UMVUE of τ.

(a) The density of one observation, with μ known and variance τ unknown, is

$$f_\tau(x) = \frac{1}{\sqrt{2\pi\tau}} \exp\left\{-\frac{(x-\mu)^2}{2\tau}\right\}, \qquad x \in \mathbb{R}.$$

For an observed sample, write $q = \sum_i (x_i - \mu)^2$. Independence gives

$$L(\tau; \underline{x}) = \prod_{i=1}^{n} f_\tau(x_i) = (2\pi\tau)^{-n/2} \exp\left\{-\frac{q}{2\tau}\right\}.$$

Taking logarithms gives

$$\ell(\tau; \underline{x}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\tau - \frac{q}{2\tau}.$$

Differentiating with respect to the variance τ gives the score and second derivative:

$$\ell'(\tau; \underline{x}) = -\frac{n}{2\tau} + \frac{q}{2\tau^2},$$
$$\ell''(\tau; \underline{x}) = \frac{n}{2\tau^2} - \frac{q}{\tau^3}.$$

For $Q = \sum_i (X_i - \mu)^2$, we have $\mathbb{E}_\tau[Q] = n\tau$. Hence

$$\begin{aligned} I_{\underline{X}}(\tau) &= -\mathbb{E}_\tau[\ell''(\tau; \underline{X})] \\ &= -\frac{n}{2\tau^2} + \frac{\mathbb{E}_\tau[Q]}{\tau^3} \\ &= -\frac{n}{2\tau^2} + \frac{n\tau}{\tau^3} = \frac{n}{2\tau^2}. \end{aligned}$$

(b) Let $Z_i = (X_i - \mu)/\sqrt{\tau} \sim \mathcal{N}(0,1)$. Then $\mathbb{E}_\tau[(X_i - \mu)^2] = \tau$ and, by the hint,

$$\mathbb{E}_\tau[(X_i - \mu)^4] = \tau^2 \mathbb{E}[Z_i^4] = 3\tau^2.$$

Thus $\mathrm{Var}_\tau((X_i - \mu)^2) = 3\tau^2 - \tau^2 = 2\tau^2$. Independence gives

$$\mathbb{E}_\tau[U] = \tau, \qquad \mathrm{Var}_\tau(U) = \frac{1}{n^2}\sum_i 2\tau^2 = \frac{2\tau^2}{n}.$$

(c) The bound for unbiased estimation of τ is $1/I_{\underline{X}}(\tau) = 2\tau^2/n$. The unbiased estimator U attains it for every $\tau > 0$, so it is the UMVUE.

5. Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathsf{Gamma}(\alpha, 1/\theta)$, where the shape $\alpha > 0$ is known, $\theta > 0$ is an unknown scale parameter, and $n\alpha > 2$. Thus

$$f_\theta(x) = \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha}, \qquad x > 0.$$

Let $S = \sum_i X_i$. You may use the following result from HW4, Problem 5(d):

$$\mathbb{E}_\theta[S^{-1}] = \frac{1}{(n\alpha-1)\theta}, \quad \mathbb{E}_\theta[S^{-2}] = \frac{1}{(n\alpha-1)(n\alpha-2)\theta^2}.$$

(a) Compute $I_{\underline{X}}(\theta)$.

(b) In HW4, Problem 5(d), you showed $V = (n\alpha-1)/S$ is the UMVUE of the rate $g(\theta) = 1/\theta$. Now, compute $\mathrm{eff}(V)$ and show V is asymptotically efficient.

(a) For $x > 0$, the density is

$$f_\theta(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^\alpha}\exp\Big\{-\frac{x}{\theta}\Big\}.$$

For $x_1, \ldots, x_n > 0$, write $s = \sum_i x_i$. Independence gives the likelihood

$$L(\theta; \underline{x}) = \prod_{i=1}^n f_\theta(x_i) = \frac{\prod_{i=1}^n x_i^{\alpha-1}}{\Gamma(\alpha)^n\theta^{n\alpha}}\exp\Big\{-\frac{s}{\theta}\Big\}.$$

Taking logarithms gives

$$\ell(\theta; \underline{x}) = (\alpha-1)\sum_{i=1}^n \log x_i - n\log\Gamma(\alpha) - n\alpha\log\theta - \frac{s}{\theta}.$$

The first two terms do not depend on θ, since α is known. Therefore the score and second derivative are

$$\ell'(\theta; \underline{x}) = -\frac{n\alpha}{\theta} + \frac{s}{\theta^2},$$
$$\ell''(\theta; \underline{x}) = \frac{n\alpha}{\theta^2} - \frac{2s}{\theta^3}.$$

Since $\mathbb{E}_\theta[X_i] = \alpha\theta$, we have $\mathbb{E}_\theta[S] = n\alpha\theta$, and

$$\begin{aligned} I_{\underline{X}}(\theta) &= -\mathbb{E}_\theta[\ell''(\theta; \underline{X})] \\ &= -\frac{n\alpha}{\theta^2} + \frac{2\mathbb{E}_\theta[S]}{\theta^3} \\ &= -\frac{n\alpha}{\theta^2} + \frac{2n\alpha\theta}{\theta^3} = \frac{n\alpha}{\theta^2}. \end{aligned}$$

(b) The supplied inverse moments give $\mathbb{E}_\theta[V] = 1/\theta$ and

$$\mathrm{Var}_\theta(V) = \frac{(n\alpha-1)^2}{(n\alpha-1)(n\alpha-2)\theta^2} - \frac{1}{\theta^2} = \frac{1}{(n\alpha-2)\theta^2}.$$

For $g(\theta) = 1/\theta$, we have $g'(\theta) = -1/\theta^2$, so

$$\ell_{CR}(\theta) = \frac{1/\theta^4}{n\alpha/\theta^2} = \frac{1}{n\alpha\theta^2}.$$

Therefore

$$\mathrm{eff}_\theta(V) = \frac{1/(n\alpha\theta^2)}{1/((n\alpha-2)\theta^2)} = \frac{n\alpha-2}{n\alpha} = 1 - \frac{2}{n\alpha}.$$

For fixed $\alpha > 0$ and every $\theta > 0$, this ratio tends to one as $n \to \infty$. Thus V is asymptotically efficient in the efficiency-ratio sense of Lecture 13. It is a UMVUE that does not attain the bound for any finite $n\alpha > 2$.