

STA 332/MATH 343: Homework 5

Duke University, Fall 2026

Due: Monday, October 19 at 11:59 PM on Gradescope

Fisher Information and the Cramér–Rao Lower Bound

- Concepts relating to Fisher information in a regular one-parameter model:
 - Define the score function.
 - Give the two formulas for Fisher information.
 - State the Cramér–Rao lower bound for an unbiased estimator of $g(\theta)$ based on n i.i.d. observations.
 - Define $\text{eff}(T)$ for an unbiased estimator T , and what it means for an unbiased estimator to be efficient.
 - Explain why an unbiased efficient estimator is automatically the UMVUE.
- Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ is unknown and $\sigma^2 > 0$ is known.
 - Compute the score for one observation and verify that its mean is zero.
 - Compute $I_{\underline{X}}(\mu)$, i.e. the Fisher information of $\underline{X}$.
 - Compute the mean and variance of $T = \bar{X}_n$.
 - Use the Cramér–Rao bound to prove that T is the UMVUE of μ .
 - Without doing any work, what is $I_T(\mu)$ and why?
- Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Ber}(\theta)$, where $0 < \theta < 1$.
 - Compute $I_{\underline{X}}(\theta)$, i.e. the Fisher information of $\underline{X}$.
 - For $T = \bar{X}_n$, compute $\mathbb{E}_\theta[T]$ and $\text{Var}_\theta(T)$.
 - Use the Cramér–Rao bound to prove that T is the UMVUE of θ .
- Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \tau)$, where μ is known and the variance $\tau > 0$ is unknown.
 - Compute $I_{\underline{X}}(\tau)$.
 - For $U = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2$, compute $\mathbb{E}_\tau[U]$ and $\text{Var}_\tau(U)$.
Hint: if $Z \sim \mathcal{N}(0, 1)$ then $\mathbb{E}[Z^4] = 3$.
 - Use the Cramér–Rao bound to prove that U is the UMVUE of τ .
- Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Gamma}(\alpha, 1/\theta)$, where the shape $\alpha > 0$ is known, $\theta > 0$ is an unknown scale parameter, and $n\alpha > 2$. Thus

$$f_\theta(x) = \frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha) \theta^\alpha}, \quad x > 0.$$

Let $S = \sum_i X_i$. You may use the following result from HW4, Problem 5(d):

$$\mathbb{E}_\theta[S^{-1}] = \frac{1}{(n\alpha - 1)\theta}, \quad \mathbb{E}_\theta[S^{-2}] = \frac{1}{(n\alpha - 1)(n\alpha - 2)\theta^2}.$$

- (a) Compute $I_{\underline{X}}(\theta)$.
- (b) In HW4, Problem 5(d), you showed $V = (n\alpha - 1)/S$ is the UMVUE of the rate $g(\theta) = 1/\theta$. Now, compute $\text{eff}(V)$ and show V is asymptotically efficient.