

STA 332/MATH 343: Homework 6

Duke University, Fall 2026

Due: Thursday, October 29 at 11:59 PM on Gradescope

Hypothesis Testing

When asked for an implementable test, specify as simple a rejection statistic as possible and its cutoff.

1. Definitions

- (a) Define the power function, size, and level of a test.
- (b) Express the Type I and Type II error probabilities at a given θ in terms of the power function.
- (c) Define a uniformly most powerful (UMP) level- α test.
- (d) Define the monotone likelihood ratio (MLR) property.

2. Testing a uniform endpoint. Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Unif}[0, \theta]$, where $\theta > 0$. We wish to test

$$H_0 : \theta \leq 1 \quad \text{versus} \quad H_1 : \theta > 1.$$

Let $M = \max\{X_1, \dots, X_n\}$ and fix an $\alpha \in (0, 1)$.

- (a) Consider the test that rejects when $M > c_\alpha$. Find c_α so that the test has size exactly α .
- (b) Compute the power function for all $\theta > 0$.
- (c) Now consider the test which rejects when $M > 1$. What is its size? Compute its power function for all $\theta > 0$.
- (d) Plot the power functions for the two tests on the same axes and compare. Is one test uniformly more powerful than the other?

3. Power of a two-sided normal test. Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, 1)$, where $\theta \in \mathbb{R}$. For

$$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta \neq 0,$$

consider the test

$$\psi(\underline{X}) = \mathbb{1}_{\{|\sqrt{n} \bar{X}_n| > z_{\alpha/2}\}}.$$

Here, we use the definition $z_\alpha = \Phi^{-1}(1 - \alpha)$, where Φ is the standard normal cdf.

- (a) Verify that this test has level α . Does it have size exactly α ?
- (b) Find its power function $\beta(\theta)$ in terms of Φ .
- (c) For each fixed $\theta \in \mathbb{R}$, compute $\lim_{n \rightarrow \infty} \beta(\theta)$.

4. **Testing a normal variance.** Let $X_1, \dots, X_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$, where μ is known and $\sigma^2 > 0$ is unknown. Let $T = \sum_{i=1}^n (X_i - \mu)^2$.

(a) By standardizing, show that

$$\frac{T}{\sigma^2} = \sum_{i=1}^n Z_i^2$$

for suitable i.i.d. standard Gaussian random variables $Z_1, \dots, Z_n$.

A sum of squares of n i.i.d. standard Gaussian random variables is called a *chi-squared random variable with n degrees of freedom*, denoted χ_n^2 . Write F_n for its cdf and $\chi_{n,\alpha}^2 = F_n^{-1}(\alpha)$ for its *lower* α -quantile (in contrast to our convention $z_\alpha = \Phi^{-1}(1 - \alpha)$ for Gaussian quantiles).

- (b) For $\sigma_1^2 > \sigma_0^2 > 0$, use Neyman–Pearson to find the implementable MP level- α test for $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 = \sigma_1^2$.
- (c) Show that the family has MLR in the sufficient statistic T . Use Karlin–Rubin to find the UMP level- α test for $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_1 : \sigma^2 > \sigma_0^2$.
- (d) Find the power function for the test in part (c), for all $\sigma^2 > 0$. Verify from the power function that the size of the test is α .

5. **A logistic location model.** Suppose we are given a single sample X_1 with density

$$f_\theta(x) = \frac{e^{x-\theta}}{(1 + e^{x-\theta})^2}, \quad x \in \mathbb{R}, \quad \theta \in \mathbb{R}.$$

You may use $\mathbb{P}_\theta(X > c) = 1/(1 + e^{c-\theta})$.

- (a) Prove that $h(x) = (1 + e^x)/(1 + e^{x-1})$ is increasing in x .
- (b) Find the implementable MP level- α test for $H_0 : \theta = 0$ versus $H_1 : \theta = 1$.
- (c) Find the power function of this test and verify that $\beta(0) = \alpha$.

6. **A lower-tail Karlin–Rubin test.** Let $X_1, \dots, X_n$ be i.i.d. with density

$$f_\theta(x) = \frac{2\sqrt{\theta}}{\sqrt{\pi}} e^{-\theta x^2}, \quad x > 0, \quad \theta > 0,$$

and let $T = \sum_{i=1}^n X_i^2$.

- (a) Show that T is sufficient. For $\theta_1 > \theta_0$, compute the likelihood ratio and determine whether it is increasing or decreasing in T . Identify a statistic in which the family has nondecreasing MLR.
- (b) By comparing densities, show that $\sqrt{2\theta} X_i$ has the same distribution as $|Z|$, where $Z \sim \mathcal{N}(0, 1)$. Deduce the distribution of $2\theta T$.
- (c) Use Karlin–Rubin to find the implementable UMP level- α test for

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

Formulate the test using the χ^2 quantiles introduced in Problem 4.

- (d) Find the power function and verify directly that the test has size α . Explain why small values of T favor the alternative.