STA 332/MATH 343: Homework 7

Duke University, Fall 2026

Due: Monday, November 9 at 11:59 PM on Gradescope
You may use software to evaluate distribution functions and quantiles, and to solve the equation in Problem 5(e) numerically.

1. **Joint MLE of a normal mean and variance.** *This problem walks you through how to find the joint MLE* $\arg\max_{\mu\in\mathbb{R},\sigma^2>0} L_{\underline{x}}(\mu,\sigma^2)$ *for normal data without needing any Calc III skills (no setting gradient to zero and checking the Hessian is negative definite).*

   Let $X_1,\dots,X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu,\sigma^2)$, where both $\mu\in\mathbb{R}$ and $\sigma^2>0$ are unknown. Set $Q(\mu)=\sum_{i=1}^n (x_i-\mu)^2$.

   (a) Write the likelihood $L_{\underline{x}}(\mu,v)$ and log likelihood, using $v=\sigma^2$ as the variance parameter.

   (b) Let $\widehat{v}(\mu)=\arg\max_{v>0} L_{\underline{x}}(\mu,v)$ be the MLE for v when μ is fixed. Find $\hat{v}(\mu)$ and use the second-derivative check to verify it really is a maximum.

   (c) Prove $Q(\mu)=Q(\bar{x})+n(\bar{x}-\mu)^2$. Use this to show that for every $v>0$,

   $$\bar{x}=\arg\max_{\mu\in\mathbb{R}} L_{\underline{x}}(\mu,v).$$

   Use the second-derivative check to verify $\bar{x}$ really is a maximum, regardless of the value of v.

   (d) Combine (b) and (c) to find the joint MLE $(\widehat{\mu},\widehat{v})=\arg\max_{\mu\in\mathbb{R},v>0} L_{\underline{x}}(\mu,v)$. To show this, first justify the inequality below, and explain why it confirms the joint MLE is what it is.

   $$L_{\underline{x}}(\bar{x},\widehat{v}(\bar{x}))\geq L_{\underline{x}}(\bar{x},v)\geq L_{\underline{x}}(\mu,v)\qquad \forall\mu\in\mathbb{R},v>0$$

2. **The normal LRT and its p-value.** *This problem uses Problem 1 to show that the normal LRT with unknown variance is just the t-test.*

   Continue with the model and assumptions of Problem 1, and test

   $$H_0:\mu=\mu_0\qquad\text{versus}\qquad H_1:\mu\neq\mu_0,$$

   with σ^2 unknown. Define $s^2=(n-1)^{-1}\sum_{i=1}^n(x_i-\bar{x})^2$ for a given $\underline{x}$, and $S^2=(n-1)^{-1}\sum_{i=1}^n(X_i-\bar{X}_n)^2$, the corresponding random variable.

   (a) Use Problem 1 to compute

   $$\Lambda(\underline{x})=\frac{\sup_{v>0} L_{\underline{x}}(\mu_0,v)}{\sup_{\mu\in\mathbb{R},\,v>0} L_{\underline{x}}(\mu,v)}.$$

   (b) Express $\Lambda(\underline{x})$ as a function of $t=\sqrt{n}(\bar{x}-\mu_0)/s$. Show that small values of Λ correspond to large values of $|t|$.

   (c) You may use that $T=\sqrt{n}(\bar{X}-\mu_0)/S\sim t_{n-1}$ under H_0, for every $\sigma^2>0$. Give the exact level-α LRT and show it coincides with the t-test we introduced in lecture.

   (d) For $n=16$, $\mu_0=10$, $\bar{x}=11.2$, and $s=2$, compute the p-value and state whether or not you would reject the null at levels 5% and 1%.

3. **More normal tests and p-values.** Let $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, 1)$.

   (a) Consider the hypotheses $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. For a given α, work out the cutoff c such that the two-sided z-test with rejection region $\mathcal{R} = \{\underline{x} : |\bar{x}| \geq c\}$ has size α.

   (b) Suppose $n = 80$ and $\bar{x} = 0.24$. Compute the p-value for the two-sided z-test from part (a). Would you reject the null at level 5%? What about level 1%?

   (c) Consider the hypotheses $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$. For a given α, work out the cutoff c such that the one-sided z-test with rejection region $\mathcal{R} = \{\underline{x} : \bar{x} \geq c\}$ has size α.

   (d) Suppose $n = 80$ and $\bar{x} = 0.24$. Compute the p-value for the one-sided z-test from part (c). Would you reject the null at level 5%? What about level 1%?

4. **p-values for testing the endpoint of a uniform distribution.** Recall the HW6 model $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \mathsf{Unif}[0, \theta]$ and hypotheses $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$. Consider a test which at level α has rejection region of the form $\mathcal{R} = \{\underline{x} : \max(x_1, \dots, x_n) \geq c_\alpha\}$.

   (a) Suppose $n = 20$ and we observe $\max(x_1, \dots, x_n) = 0.97$. Compute the p-value.

   (b) Suppose $n = 20$ and we observe $\max(x_1, \dots, x_n) = 1.03$. Compute the p-value and interpret it.

5. **An LRT beyond the normal model.** Let $X_1, \dots, X_n$ be i.i.d. with density

$$f_\theta(x) = r\theta x^{r-1} e^{-\theta x^r}, \qquad x > 0,$$

   where $r > 0$ is known and $\theta > 0$ is unknown. Set $T = \sum_{i=1}^n X_i^r$ and consider testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$.

   (a) Find the MLE of θ and derive the likelihood ratio statistic. Show that it can be written as

$$\Lambda(\underline{x}) = h(T/n), \qquad h(u) := u^n e^{n(1-u)}.$$

   Conclude that the LRT rejects when $h(T/n) < c$, where $c = c_\alpha$ is chosen to ensure the test has desired size α.

   (b) Show that h is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. Conclude that the LRT has the form: reject when $T/n < a_c$ or $T/n > b_c$, where $a_c \leq 1$, $b_c \geq 1$, $h(a_c) = h(b_c) = c$.

   (c) Show that $X_i^r \sim \text{Exp}(1)$ under the null distribution.

   (d) You may use that the sum of n i.i.d. Exp(1)'s is Gamma$(n, 1)$. Let $F_n(t)$ be the cdf of Gamma$(n, 1)$, meaning $F_n(t) = \mathbb{P}(\text{Gamma}(n, 1) \leq t)$ for all $t \geq 0$. Write an equation in terms of F_n that must be satisfied to ensure the LRT has size exactly α.

   (e) Suppose $n = 1$ and the observed value of $T/n = T$ is 2. Compute the exact LRT p-value using its definition as the smallest level α at which we would reject the null.

   Guidance: α decreases $\iff$ $c = c_\alpha$ decreases $\iff$ a_c decreases and b_c increases. Therefore the smallest possible α at which we would reject corresponds to the smallest a_c and largest b_c at which we would reject. Conclude the largest b_c is precisely $b_c = 2 = T$, corresponding to $c = h(b_c) = h(2)$. What is the corresponding a_c? Find it numerically, then compute the p-value as

$$p = \mathbb{P}_{\theta=1}(T < a_c) + \mathbb{P}_{\theta=1}(T > b_c)$$

   for the above a_c and b_c.